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Coherent states of a relativistic particle in an external electromagnetic field

V G Bagrov, I L Buchbinder and D M Gitman

Department of Mathematical Analysis, Pedagogical Institute, State University, Tomsk, USSR

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Abstract. The problem of the construction of coherent states of relativistic particles (bosons and fermions) in electromagnetic fields has been solved. The states discovered are exact solutions to the Klein–Gordon and Dirac equations; the wavefunctions of the particles are found in the homogeneous magnetic and electric fields and the Redmond configuration field, as well as in the plane-wave field. The main point is the use of the formulation of relativistic quantum mechanics on the 'null' plane.

1. Introduction

The coherent states of the harmonic oscillator were discovered by Schrödinger (1926), who pointed out that these states reduce the uncertainty relations for coordinates and momentum to a minimum. Special attention has to be paid to these states since the publication of Glauber's work (Glauber 1963a,b). He pointed out that the coherent states of an electromagnetic field are especially useful in different problems in quantum optics. It is interesting to note that the coherent states of an electromagnetic field were introduced by Rashewski (1958), but his work is unknown to physicists.

At the present time, it is known that coherent states can be constructed not only for oscillator (electromagnetic field), but for arbitrary physical systems as well. Malkin and Man'ko introduced the method of construction of coherent states for non-relativistic particles (Malkin and Man'ko 1968, 1970, 1971, Malkin *et al* 1970, 1973, Dodonov *et al* 1975). These authors found the wavefunctions of the coherent states of a non-relativistic electron interacting with different external fields, and thus calculated the Green functions. The main point of the Malkin–Man'ko method is the construction of annihilation operators (integrals of motion) based on the evolution operator of the system considered.

In applying the Malkin–Man'ko method to the construction of coherent states of relativistic particles, we come across several problems. The method cannot be applied to the Klein–Gordon equation because of the absence of an evolution operator, and in the case of the Dirac equation several calculating difficulties arise. In our previous paper (Bagrov *et al* 1975) we suggest a new approach to the construction of the coherent states of relativistic quantum theory on the 'null' plane (Kogut and Soper 1970, Bjorken *et al* 1971, Rohrlich 1970, Neville and Rohrlich 1971, Chang *et al* 1973, Chang and Yan 1973). In the variables of the 'null' plane, the Klein–Gordon and Dirac equations are first-order equations dependent on some 'time'. The evolution operator can be introduced easily. In this paper we have constructed coherent states of relativistic particles in homogeneous magnetic and electric fields and in the Redmond configuration field (Redmond 1965) on the one hand, and in the plane-wave field on the other. These coherent states, which are exact solutions of the Klein–Gordon and Dirac equations, are of great importance in the calculation of various physical effects. For example, the Volkov solution of the Dirac equation in the plane-wave field (Volkov 1935) is effectively used in relativistic quantum mechanics (see for example Nikishov and Ritus 1964a, b, 1967, Ternov *et al* 1968). We hope that our new solutions will also be useful.

2. The main points of relativistic quantum mechanics on the 'null' plane

In the construction of coherent states of relativistic particles we shall use the formulation of relativistic quantum mechanics on the 'null' plane. The coordinates of the 'null' plane u^μ are:

$$u^0=\frac{x^0-x^3}{\sqrt{2}},\qquad u^1=x^1,\qquad u^2=x^2,\qquad u^3=\frac{x^0+x^3}{\sqrt{2}}. \tag{1}$$

On the null plane the Klein–Gordon equation is:

$$\begin{aligned} &\mathrm{i}\hbar(\partial\psi/\partial u^0)=\mathscr{H}_{\mathrm{K}}\psi \\ &\mathscr{H}_{\mathrm{K}}=\frac{e}{c}\tilde{A}_0+\tfrac{1}{2}\tilde{\mathscr{P}}_3^{-1}\left(\tilde{\mathscr{P}}_1^2+\tilde{\mathscr{P}}_2^2+m^2c^2+\mathrm{i}\frac{e\hbar}{c}\tilde{F}_{03}\right) \end{aligned} \tag{2}$$

where

$$\tilde{F}_{\alpha\beta}=\frac{\partial\tilde{A}_\beta}{\partial u^\alpha}-\frac{\partial\tilde{A}_\alpha}{\partial u^\beta}$$

$$\tilde{\mathscr{P}}_\alpha=\mathrm{i}\hbar\frac{\partial}{\partial u^\alpha}-\frac{e}{c}\tilde{A}_\alpha$$

and $\tilde{A}_\alpha$ is the potential of the external field in the coordinate system connected with the coordinates u^μ (equation (1)). The scalar product for the Klein–Gordon equation on the 'null' plane is defined as:

$$(\psi,\phi)_{u^0}=\int \mathrm{d}u\,[\psi^*(u)\tilde{\mathscr{P}}_3\phi(u)+\tilde{\mathscr{P}}_3\psi(u))^*\phi(u)]. \tag{3}$$

Further, we will need an explicit expression for the coordinate operator for the particle, described by the Klein–Gordon equation on the 'null' plane. It is obvious that the multiplication operator u^n ($n=1,2,3$) is not the Hermitian operator corresponding to the scalar product (3). However, we can construct the Hermitian operator for the scalar product (3) with all the properties of the coordinate operator. This operator is:

$$q^n=u^n-\tfrac{1}{2}\delta^{n,3}(\partial/\partial u^3)^{-1}\qquad (n=1,2,3). \tag{4}$$

The operator (4) is the Newton–Wigner coordinate operator (Newton and Wigner 1949) on the 'null' plane and is constructed as the Hermitian part of the operator u^n

corresponding to the scalar product (3). The Dirac equation on the 'null' plane is

$$
\begin{aligned}
&\mathrm{i}\hbar(\partial\psi_{(-)}/\partial u^0)=\mathscr{H}_{\mathrm{D}}\psi_{(-)}\\
&\psi_{(-)}=P_{(-)}\Psi, \qquad P_{(-)}=\tfrac{1}{2}\tilde{\gamma}^3\tilde{\gamma}^0\\
&\mathscr{H}_{\mathrm{D}}=\frac{e}{c}\tilde{A}_0+\frac{1}{4}(\tilde{\gamma}^j\tilde{\mathscr{P}}_j+mc)\tilde{\gamma}^3\tilde{\mathscr{P}}_3^{-1}(\tilde{\gamma}^j\tilde{\mathscr{P}}_j+mc)\tilde{\gamma}^0 \qquad (j=1,2)\\
&\tilde{\gamma}^0=\frac{\gamma^0-\gamma^3}{\sqrt{2}}, \qquad \tilde{\gamma}^1=\gamma^1, \qquad \tilde{\gamma}^2=\gamma^2, \qquad \tilde{\gamma}^3=\frac{\gamma^0+\gamma^3}{\sqrt{2}}.
\end{aligned} \tag{5}
$$

The scalar product is expressed as:

$$(\psi_{(-)},\phi_{(-)})=\sqrt{2}\int \mathrm{d}u\,\psi^{\dagger}_{(-)}\phi_{(-)}. \tag{6}$$

Since the Klein–Gordon and Dirac equations are first-order equations in 'time' u^0, we can introduce the evolution operator $U(u^0, u^{0\prime})$ in a general way. With the evolution operator $U(u^0, u^{0\prime})$ we can construct the operator integral of motion:

$$I(u^0)=U(u^0,0)IU^{-1}(u^0,0)$$

when $I=a$, where a is the annihilation operator, we shall have an annihilation operator integral of motion.

3. Coherent states of relativistic particles in an electric field and the Redmond configuration field

We consider the construction of coherent states for equation (2) where the potentials A_α are:

$$
\begin{aligned}
&\tilde{A}_0=0, \qquad \tilde{A}_1=\frac{c\hbar}{e}f_1(u^0)+\frac{H}{2}u^2,\\
&\tilde{A}_2=-\frac{c\hbar}{e}f_2(u^0)-\frac{H}{2}u^1, \qquad \tilde{A}_3=\frac{c\hbar}{e}f_0(u^0)
\end{aligned} \tag{7}
$$

where $f_i=f_i(u^0)(i=0,1,2)$ are arbitrary functions of u^0. Such a choice of potentials corresponds to the electric field along the x^3 axis and the Redmond configuration field. Equation (2) with potentials (7) becomes:

$$
\begin{aligned}
\mathrm{i}\hbar\frac{\partial\psi}{\partial u^0}=\frac{1}{2}\left(\mathrm{i}\hbar\frac{\partial}{\partial u^3}-\hbar f_0\right)^{-1}&\left[\left(\mathrm{i}\hbar\frac{\partial}{\partial u^1}-\hbar f_1-\frac{eH}{2c}u^2\right)^2\right.\\
&\left.+\left(\mathrm{i}\hbar\frac{\partial}{\partial u^2}+\hbar f_2+\frac{eH}{2c}u^1\right)^2+m^2c^2+\mathrm{i}\hbar^2 f_0'\right]\Psi
\end{aligned} \tag{8}
$$

$$f_0'=\mathrm{d}f_0/\mathrm{d}u^0.$$

The operator $\mathrm{i}(\partial/\partial u^3)$ is the integral of motion for equation (8). The wavefunction Ψ can be chosen as the eigenfunction for this operator:

$$
\begin{aligned}
&\mathrm{i}(\partial\psi/\partial u^3)=k_3\Psi\\
&\psi_{k_3}(u)=\mathrm{e}^{-\mathrm{i}k_3u^3}\phi(u^0,u^1,u^2).
\end{aligned} \tag{9}
$$

For the function ϕ from equations (8), (9) we get:

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial u^0}=\frac{1}{2\hbar(k_3-f_0)}\bigg[\Big(\mathrm{i}\hbar\frac{\partial}{\partial u^1}-\hbar f_1-\frac{eH}{2c}u^2\Big)^2 + (\mathrm{i}\hbar\frac{\partial}{\partial u^2}+\hbar f_2+\frac{eH}{2c}u^1\Big)^2+m^2c^2+\mathrm{i}\hbar^2 f_0'\bigg]\phi. \tag{10}$$

We introduce the Bose annihilation and creation operators:

$$\begin{aligned}
c_1&=\frac{1}{\sqrt{2\gamma}}\bigg[\frac{\partial}{\partial u^1}+\frac{\gamma}{2}u^1+\mathrm{i}\epsilon\Big(\frac{\partial}{\partial u^2}+\frac{\gamma}{2}u^2\Big)\bigg]\\
c_2&=\frac{1}{\sqrt{2\gamma}}\bigg[\frac{\partial}{\partial u^1}+\frac{\gamma}{2}u^1-\mathrm{i}\epsilon\Big(\frac{\partial}{\partial u^2}+\frac{\gamma}{2}u^2\Big)\bigg]\\
c_1^\dagger&=\frac{1}{\sqrt{2\gamma}}\bigg[-\frac{\partial}{\partial u^1}+\frac{\gamma}{2}u^1-\mathrm{i}\epsilon\Big(-\frac{\partial}{\partial u^2}+\frac{\gamma}{2}u^2\Big)\bigg]\\
c_2^\dagger&=\frac{1}{\sqrt{2\gamma}}\bigg[-\frac{\partial}{\partial u^1}+\frac{\gamma}{2}u^1+\mathrm{i}\epsilon\Big(-\frac{\partial}{\partial u^2}+\frac{\gamma}{2}u^2\Big)\bigg]\\
\gamma&=|e|H/c\hbar, \qquad \epsilon=\operatorname{sgn} e.
\end{aligned} \tag{11}$$

We now construct the annihilation operator integrals of motion:

$$C_n(t)=U(t,0)c_nU^{-1}(t,0) \qquad (n=1,2) \tag{12}$$

where $U(t,0)$ is the evolution operator for equation (10):

$$U(t,0)=\exp\bigg\{-\frac{\mathrm{i}}{2\hbar^2}\int_0^t \mathrm{d}t\bigg[\Big(\mathrm{i}\hbar\frac{\partial}{\partial u^1}-\hbar f_1(t)-\frac{\epsilon\gamma}{2}u^2\Big)^2 +\Big(\mathrm{i}\hbar\frac{\partial}{\partial u^2}+\hbar f_2(t)+\frac{\epsilon\gamma}{2}u^1\Big)^2+m^2c^2+\mathrm{i}\hbar^2 f_0'(t)\bigg]\bigg\} \tag{13}$$

$$t=t(u^0)=\int\frac{\mathrm{d}u^0}{k_3-f_0(u^0)}.$$

The operators $C_n(t)$ can easily be found:

$$\begin{aligned}
C_1(t)&=c_1\,\mathrm{e}^{\mathrm{i}\gamma t}+Q(t)\sqrt{\gamma/2}\\
C_2(t)&=c_2\\
Q(t)&=\int_0^t \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\gamma t}(\mathrm{i}\epsilon f_2(t)-f_1(t)).
\end{aligned} \tag{14}$$

Let us suppose the function ϕ is an eigenfunction for the operators $C_n(t)(n=1,2)$:

$$C_n(t)\phi=\frac{z_n}{\sqrt{2}}\phi. \tag{15}$$

The wavefunction, which is the solution of equations (9), (10), (15) and the normalized relative scalar product (3), is given by

$$\psi_{k_3z_1z_2}(u)=N_0q^{-1/2}\,\mathrm{e}^{\chi(t)} \tag{16}$$

$$q=k_3-f_0$$

$$4\chi=2z_1z_2(1-\mathrm{e}^{-\mathrm{i}\gamma t})+2\mathrm{i}(\gamma+k_0^2)t-\gamma[(u^1)^2+(u^2)^2]-4\mathrm{i}k_3u^3+2(\mu u^1-\mathrm{i}\epsilon\nu u^2)$$
$$+2z_2\sqrt{\gamma}Q\,\mathrm{e}^{-\mathrm{i}\gamma t}+2z_1\sqrt{\gamma}Q^*-2\gamma\int_0^t QQ'^*\,\mathrm{d}t-2\mathrm{i}\int_0^t(f_1^2+f_2^2)\,\mathrm{d}t \tag{17}$$

$$\mu=\nu+2\sqrt{\gamma}z_2=(\sqrt{\gamma}z_1-\gamma Q)\,\mathrm{e}^{-\mathrm{i}\gamma t}+\sqrt{\gamma}z_2$$
$$\gamma N_0^{-2}=8\pi^2\hbar\exp[\tfrac{1}{2}(|z_1|^2+|z_2|^2)+z_1z_2],\qquad k_0=mc/\hbar. \tag{18}$$

The wavefunctions (16) form a complete system of functions with the following complete condition:

$$2\gamma\int \mathrm{d}k_3\int \mathrm{d}^2z_1\int \mathrm{d}^2z_2\psi^*_{k_3z_1z_2}(u',u^0)\mathscr{P}_3\psi_{k_3z_1z_2}(u,u^0)=\pi\delta(u-u'). \tag{19}$$

On the basis of the functions (16) we can calculate the average values of coordinates $u^1=x$ and $u^2=y$:

$$\overline{x(t)}+\mathrm{i}\epsilon\overline{y(t)}=\frac{1}{\sqrt{\gamma}}(z_1-\sqrt{\gamma}Q(t))\,\mathrm{e}^{-\mathrm{i}\gamma t}+\frac{1}{\sqrt{\gamma}}z_2^*. \tag{20}$$

The average value (20) coincides exactly with the trajectory obtained from the classical relativistic equations for such a case.

For the coordinates u^1, u^2 and corresponding momenta, the function (16) gives a minimum for the uncertainty relations.

It is interesting to note that when $\gamma\to 0$ and $f_0\to 0$ the function (16) is transformed into the Volkov solution (Volkov 1935) (to an exactness of N_0) for the particle without spin in the plane-wave field. However, the trajectory loses its meaning.

In the pure homogeneous magnetic field it is necessary to insert $f_i=0 (i=0,1,2)$ in the expressions (16)–(20). In this special case the coherent states for the Klein–Gordon equation were found in Dodonov *et al* (1975), without using the 'null' plane method.

The trajectory in a homogeneous magnetic field has a simple form:

$$\overline{x(t)}=\frac{1}{\sqrt{\gamma}}(|z_1|\cos\delta+\mathrm{Re}\,z_2)$$
$$\overline{y(t)}=-\frac{\epsilon}{\sqrt{\gamma}}(|z_1|\sin\delta-\mathrm{Im}\,z_2) \tag{21}$$
$$\delta=\gamma t-\phi_0,\qquad z_1=|z_1|\,\mathrm{e}^{\mathrm{i}\phi_0}$$

describing the motion along a circle in the x, y plane.

The solution of the Dirac equation in the external field with the potentials (7) is obtained from the function (16) as a result of the action of the special matrix operator:

$$\psi_D = N_q^{-1} \begin{bmatrix} k_0 + \sqrt{2}q - \sigma_3(\boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{F}}) \\ (k_0 - \sqrt{2}q)\sigma_3 - (\boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{F}}) \end{bmatrix} e^{\frac{1}{2}i\epsilon\gamma t + \chi} v$$
$$\boldsymbol{\mathcal{F}} = \boldsymbol{e}_1[i\mu - \epsilon\gamma(y + i\epsilon x) - f_1] + \boldsymbol{e}_2[\epsilon\nu - i\gamma(y + i\epsilon x) + f_2]. \tag{22}$$

$\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors on the x and y axes. The arbitrary two-component spinor v defines the spin orientation and $2(v^\dagger v)N^2 = \hbar N_0^2$. The trajectory (20) is also the same as in the case of the Dirac equation.

4. Coherent states in the plane-wave field for the Klein–Gordon equation

Let us consider the case when, in equation (2), $\tilde{A}_0 = \tilde{A}_3 = 0$, $\tilde{A}_1 = A_1(u^0)$, $\tilde{A}_2 = A_2(u^0)$. Such a choice of potentials corresponds to the plane-wave field. Assuming the operators (14) and function (16) have $\gamma = 0$ and $f_0 = 0$, we would be able to obtain all the results in this field. However, when $\gamma \to 0$, the wavefunction is transformed into the Volkov solution, which is not a coherent state. Hence to find the coherent states in the plane-wave field we must use a different definition of the annihilation operators from that in (14). In the plane-wave field the Hamiltonian $\mathcal{H}_K$ in equation (2) is independent of operators u^n $(n = 1, 2, 3)$. Therefore it is more convenient to find the solution of this equation in the momentum representation. We can write the Fourier transformation of the wavefunction:

$$\psi(\boldsymbol{u}, u^0) = \int \frac{d^3k}{(2\pi)^3} e^{-i\boldsymbol{k}\cdot\boldsymbol{u}} \psi(\boldsymbol{k}, u^0). \tag{23}$$

Equations (2), (3) and (4) in the momentum representation are:

$$i(\partial\psi(\boldsymbol{k}, u^0/\partial u^0) = \omega(\boldsymbol{k}, u^0)\psi(\boldsymbol{k}, u^0) \tag{24}$$

$$\omega(\boldsymbol{k}, u^0) = \frac{1}{2k_3}\left[\left(k_1 - \frac{e}{c\hbar}A_1\right)^2 + \left(k_2 - \frac{e}{c\hbar}A_2\right)^2 + k_0^2\right] \tag{24a}$$

$$(\psi, \phi) = 2\hbar \int d^3k\psi^*(\boldsymbol{k}, u^0)k_3\phi(\boldsymbol{k}, u^0) \tag{24b}$$

$$q^n = -i\left(\frac{\partial}{\partial k_n} + \frac{\delta^{n,3}}{2k_3}\right). \tag{24c}$$

We shall construct the coherent states for the Klein–Gordon equation in two cases: when the operator $\tilde{\mathcal{P}}_3$ is the integral of motion and when $\tilde{\mathcal{P}}_3$ is not the integral of motion.

4.1. *Operator $\tilde{\mathcal{P}}_3$ is the integral of motion*

It is easy to see that the operator $\tilde{\mathcal{P}}_3$ is the integral of motion for equation (24). In this case, the solution for equation (24) may be chosen as the eigenfunction for it. In the momentum representation $\tilde{\mathcal{P}}_3 = \hbar k_3$. We assume

$$\hbar k_3\psi = \hbar\tilde{k}_3\psi \tag{25}$$

then

$$\psi_{\tilde{k}_3} = \delta(k_3 - \tilde{k}_3)\phi(u^0, k_1, k_2). \tag{26}$$

The equation for the function ϕ is obtained from the equation (24):

$$\mathrm{i}(\partial\phi/\partial u^0) = \tilde{\omega}(\boldsymbol{k}, u^0)\phi \tag{27}$$

where

$$\tilde{\omega}(\boldsymbol{k}, u^0) = \omega(\boldsymbol{k}, u^0)|_{k_3=\tilde{k}_3}.$$

We will introduce the operators:

$$\begin{aligned} b_n &= -\frac{\mathrm{i}}{\sqrt{2}}\left(\frac{k_n}{k_0} + k_0\frac{\partial}{\partial k_n}\right) \\ b_n^\dagger &= -\frac{\mathrm{i}}{\sqrt{2}}\left(-\frac{k_n}{k_0} + k_0\frac{\partial}{\partial k_n}\right) \end{aligned} \qquad (n = 1, 2). \tag{28}$$

The operators (28) are the Bose annihilation and creation operators. Using these operators (28) we can construct the integrals of motion:

$$\begin{aligned} B_n(u^0) &= U(u^0, 0) b_n U^{-1}(u^0, 0) \\ U(u^0, 0) &= \exp\left(-\mathrm{i}\int_0^{u^0} \mathrm{d}u^0 \tilde{\omega}(\boldsymbol{k}, u^0)\right). \end{aligned} \tag{29}$$

It is easy to show that:

$$B_n(u^0) = -\frac{\mathrm{i}}{\sqrt{2}}\left(\frac{k_n}{k_0} + k_0\frac{\partial}{\partial k_n} - \mathrm{i}k_0\int_0^{u^0} \mathrm{d}u^0\frac{\partial\tilde{\omega}(\boldsymbol{k}, u^0)}{\partial k_n}\right) \qquad (n = 1, 2). \tag{30}$$

The coherent states are defined as the eigenfunctions for the operators $B_n(u^0)$:

$$B_n(u^0)\phi = \frac{z_n}{\sqrt{2}}\phi. \tag{31}$$

Equations (27) and (31) may easily be solved as a system. The wavefunction, which is the solution of equations (24), (25), (27), (31) and the normalized relative scalar product (24*b*), is:

$$\begin{aligned} \psi_{\tilde{k}_3 z_1 z_2}(\boldsymbol{k}, u^0) &= \frac{1}{k_0(2\pi\hbar\tilde{k}_3)^{1/2}}\delta(k_3 - \tilde{k}_3 \exp\left(-\mathrm{i}\int_0^{u^0} \mathrm{d}u^0\tilde{\omega}(\boldsymbol{k}, u^0)\right) \\ &\quad \times \prod_{n=1}^{2} \exp\left(-\frac{1}{2}\mathrm{Im}^2 z_n + \mathrm{i}z_n\frac{k_n}{k_0} - \frac{1}{2}\frac{k_n^2}{k_0^2}\right). \end{aligned} \tag{32}$$

On the basis of the function (32) we will calculate the average values of momenta $p_n = \hbar k_n$ and coordinates $u^n = -\mathrm{i}(\partial/\partial k_n)(n = 1, 2)$. The result is:

$$\begin{aligned} \overline{p_n(u^0)} &= -\hbar k_0 \,\mathrm{Im}\, z_n \\ \overline{u^n(u^0)} &= \frac{1}{k_0}\mathrm{Re}\, z_n + \frac{k_0}{\tilde{k}_3}\int_0^{u^0} \mathrm{d}u^0\left(\mathrm{Im}\, z_n - \frac{e}{mc^2}A^n\right) \end{aligned} \qquad (n = 1, 2) \tag{33}$$

so Re z_n and Im z_n are the average values of momenta and coordinates when $u^0 = 0$. The second equation (33) coincides with the trajectory obtained in classical relativistic mechanics. Taking the function (32) we can calculate the values of fluctuations of the coordinates and momenta. We get:

$$\overline{(\Delta p_n(u^0))^2}\,\overline{(\Delta u^n(u^0))^2}=\frac{\hbar^2}{4}\left(1+\frac{k_0^4}{2\tilde{k}_3^2}(u^0)^2\right). \tag{34}$$

The uncertainty relation has a minimum when $u^0=0$. It is known that the coherent states, constructed for the arbitrary quadratic Hamiltonian, reduce the uncertainty relation to a minimum when the Hamiltonian can be transformed to the Hamiltonian of the harmonic oscillator as a result of action by a unitary operator (Stoler 1970, Trifonov 1974). This is impossible in the plane-wave field.

We can show that the functions (32) form a complete system of functions with a complete relation:

$$\int \mathrm{d}\tilde{k}_3\,\mathrm{d}^2z_1\,\mathrm{d}^2z_2\psi^*_{\tilde{k}_3z_1z_2}(\boldsymbol{k}',u^0)\psi_{\tilde{k}_3z_1z_2}(\boldsymbol{k},u^0)\hbar\tilde{k}_3=2\pi^2\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{35}$$

We should note that the wavefunction may be written in the coordinate representation:

$$\begin{aligned}\Psi_{\tilde{k}_3z_1z_2}(\boldsymbol{u},u^0)=&\frac{1}{k_0\sqrt{\hbar\tilde{k}_3}[1+\mathrm{i}(k_0^2/\tilde{k}_3)u^0]}\exp\left(-\mathrm{i}\tilde{k}_3u^3-\mathrm{i}\frac{k_0^2}{2\tilde{k}_3}u^0\right)\\&\times\prod_{n=1}^{2}\exp\left[-\frac{1}{2}\mathrm{Im}^2z_n-\frac{\mathrm{i}}{2\tilde{k}_3}\int_0^{u^0}\mathrm{d}u^0f_n^2(u^0)\right.\\&\left.-\frac{1}{2}\left(z_n-k_0u^n-\frac{k_0}{\tilde{k}_3}\int_0^{u^0}\mathrm{d}u^0f_n(u^0)\right)^2\left(1+\mathrm{i}\frac{k_0^2}{\tilde{k}_3}u^0\right)^{-1}\right],\end{aligned} \tag{36}$$

$$f_n=\frac{e}{mc^2}A_n.$$

4.2. Operator $\tilde{\mathscr{P}}_3$ is not the integral of motion

Suppose that the operator $\mathscr{P}_3$ is not chosen as the integral of motion. Then the Hamiltonian $\mathscr{H}_{\mathrm{K}}$ in equation (2) is not in quadratic form with respect to the operator $\tilde{\mathscr{P}}_3$ and we come across difficulties in the construction of coherent states. So far, not one paper has been published on the construction of coherent states for a non-quadratic Hamiltonian.

As before, the coherent states will be constructed in the momentum representation. We will define the Bose annihilation operator (taking into account equation (4)):

$$b_n=-\frac{\mathrm{i}}{\sqrt{2}}\left(\frac{k_n}{k_0}+k_0\frac{\partial}{\partial k_n}+\frac{\delta^{n,3}}{2k_3}k_0\right)\qquad(n=1,2,3) \tag{37}$$

and construct integrals of motion $B_n(u^0)$ in accordance with the relations (29), where

$$U(u^0,0)=\exp\left(-\mathrm{i}\int_0^{u^0}\mathrm{d}u^0\omega(\boldsymbol{k},u^0)\right)$$

then:

$$B_n(u^0)=-\frac{\mathrm{i}}{\sqrt{2}}\left(\frac{k_n}{k_0}+k_0\frac{\partial}{\partial k_n}+\mathrm{i}\frac{\delta^{n,3}}{2k_3}k_0+\mathrm{i}k_0\int_0^{u^0}\mathrm{d}u^0\frac{\partial\omega(\boldsymbol{k},u^0)}{\partial k_n}\right)\qquad(n=1,2,3). \tag{38}$$

The wavefunction, which is the solution of equations (24) and (31) with the integrals of motion $B_n(u^0)$ (equation (38)) and the normalized relative scalar product (24*b*), is:

$$\psi_{z_1z_2z_3}(\boldsymbol{k}, u^0) = \frac{1}{k_0[2\pi^{3/2}\hbar k_0\Phi(-\mathrm{Im}\, z_3)]^{1/2}}|k_3|^{-1/2} \times \exp\left(-\mathrm{i}\int_0^{u^0} \mathrm{d}u^0\omega(\boldsymbol{k}, u^0)\right) \prod_{n=1}^{3} \exp\left(-\frac{1}{2}\mathrm{Im}^2 z_n + \mathrm{i}z_n\frac{k_n}{k_0} - \frac{1}{2}\frac{k_n^2}{k_0^2}\right) \tag{39}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}t\, \mathrm{e}^{-t^2}$$

is the probability integral (Gradshteyn and Ryzhik 1971). The function (39) is the coherent state of the variables k_1, k_2, k_3; the system of functions (39) is a complete system. We can write the condition of completion as:

$$\hbar|k_3|\int \mathrm{d}F(z)\psi^*_{z_1z_2z_3}(\boldsymbol{k}', u^0)\psi_{z_1z_2z_3}(\boldsymbol{k}, u^0) = 4\pi^3\delta(\boldsymbol{k}-\boldsymbol{k}')$$
$$\mathrm{d}F(z) = \mathrm{d}^2z_1\,\mathrm{d}^2z_2\,\mathrm{d}^2z_3\Phi(-\mathrm{Im}\, z_3). \tag{40}$$

With the help of function (39) we can calculate the average values of the operators p_n and q_n when $u^0 = 0$:

$$\overline{p_n} = -\hbar k_0\left(\mathrm{Im}\, z_n + \frac{\delta^{n,3}}{\sqrt{\pi}}\frac{\mathrm{e}^{-\mathrm{Im}^2 z_3}}{\mathrm{i}\Phi(\mathrm{i}\,\mathrm{Im}\, z_3)}\right)$$
$$\overline{q^n} = \frac{1}{k_0}\mathrm{Re}\, z_n. \tag{41}$$

If $u^0 \neq 0$ then:

$$\overline{p_n(u^0)} = \overline{p_n}$$
$$\overline{q^n(u^0)} = \overline{q^n} - \int_0^{u^0} \mathrm{d}u^0\overline{\frac{\partial\omega(\boldsymbol{k}, u^0)}{\partial k_n}}. \tag{42}$$

The average value $\overline{\partial\omega(\boldsymbol{k}, u^0)/\partial k_n}$ is expressed through the integral which diverges and the trajectory loses its meaning. However, the function (39) is the exact solution of the Klein–Gordon equation in the plane-wave field and the system of these functions is a complete system of functions.

5. Coherent states for the Dirac equation in the plane-wave field

In the plane-wave field ($\tilde{A}_3 = \tilde{A}_0 = 0$) the Dirac equation (5) coincides with the Klein–Gordon equation (2). Therefore, all the results obtained for the Klein–Gordon equation in the previous sections can be used for the solution of the Dirac equation. The bispinor $\psi_{(-)}$ in the momentum representation is:

$$\psi_{(-)}(\boldsymbol{k}, u^0) = \psi(\boldsymbol{k}, u^0)P_{(-)}v \tag{43}$$

where v is an arbitrary bispinor, and the function $\psi(\boldsymbol{k}, u^0)$ is the solution of equation (24). The operator u^n is the Hermitian operator relating to the scalar product (6) and may be considered as a coordinate operator.

The function ψ in the equation (43) may be chosen as a self-function either for the operators $\tilde{\mathscr{P}}_3$ and $B_n(u^0)$ ($n = 1, 2$) (30) or the operators $B_n(u^0)$ ($n = 1, 2, 3$). The latter operators for the Dirac equation are:

$$B_n(u^0) = -\frac{\mathrm{i}}{\sqrt{2}}\left(\frac{k_n}{k_0} + k_0\frac{\partial}{\partial k_n} + \mathrm{i}k_0\int_0^{u^0} \mathrm{d}u^0\frac{\partial\omega(\boldsymbol{k}, u^0)}{\partial k_n}\right) \qquad (n = 1, 2, 3). \tag{44}$$

If the function ψ is the eigenfunction for the operators $\tilde{\mathscr{P}}_3$ and $B_n(u^0)$ ($n = 1, 2$) we will have the normalized solution of the Dirac equation:

$$\psi_{(-)\tilde{k}_3 z_1 z_2}(\boldsymbol{k}, u^0) = \frac{1}{(\sqrt{2}\pi k_0^2)^{1/2}}\exp\left(-\mathrm{i}\int_0^{u^0}\mathrm{d}u^0\tilde{\omega}(\boldsymbol{k}, u^0)\right)\delta(k_3 - \tilde{k}_3)$$
$$\times\prod_{n=1}^{2}\exp\left(-\frac{1}{2}\mathrm{Im}^2 z_n + \mathrm{i}z_n\frac{k_n}{k_0} - \frac{1}{2}\frac{k_n^2}{k_0^2}\right)P_{(-)}v \tag{45}$$

where $v^\dagger P_{(-)}v = 1$. The function (45) is a complete system, with a complete relation:

$$\sqrt{2}\int \mathrm{d}\tilde{k}_3\,\mathrm{d}^2z_1\,\mathrm{d}^2z_2\psi_{(-)\tilde{k}_3 z_1 z_2}(\boldsymbol{k}', u^0)\psi^\dagger_{(-)\tilde{k}_3 z_1 z_2}(\boldsymbol{k}, u^0) = (2\pi)^2\delta(\boldsymbol{k} - \boldsymbol{k}'). \tag{46}$$

Equations (33), defining the classical trajectory for $n = 1, 2$ and the form of the uncertainty relation (34) have the same form for the Dirac equation. The quantum numbers z_1 and z_2 have the same meaning as in the solution (32).

If the function $\psi(\boldsymbol{k}, u^0)$ is the eigenfunction for the operators $B_n(u^0)$ (equation (44)) then the normalized solution of the Dirac equation is:

$$\psi_{(-)z_1 z_2 z_3}(\boldsymbol{k}, u^0) = \frac{1}{(\sqrt{2}\pi^{3/2}k_0^3)^{1/2}}\exp\left(-\mathrm{i}\int_0^{u^0}\mathrm{d}u^0\omega(\boldsymbol{k}, u^0)\right)$$
$$\times\prod_{n=1}^{3}\exp\left(-\frac{1}{2}\mathrm{Im}^2 z_n + \mathrm{i}z_n\frac{k_n}{k_0} - \frac{1}{2}\frac{k_n^2}{k_0^2}\right)P_{(-)}v. \tag{47}$$

The average values of coordinates and momenta, which have been calculated on the basis of function (47), are given by

$$\overline{p_n(u^0)} = -\hbar k_0\,\mathrm{Im}\,z_n \qquad (n = 1, 2, 3) \tag{48}$$
$$\overline{u^n(u^0)} = \frac{1}{k_0}\mathrm{Re}\,z_n + \frac{1}{k}\int_0^{u^0}\mathrm{d}u^0\left(\mathrm{Im}\,z_n - \frac{e}{mc^2}A^n\right) \qquad (n = 1, 2)$$

where

$$K = -\tfrac{1}{2}\mathrm{e}^{\mathrm{Im}^2 z_n}\left(\int_0^{\mathrm{Im}\,z_3}\mathrm{d}t\,\mathrm{e}^{t^2}\right)^{-1}. \tag{49}$$

The expression for $\overline{u^n(u^0)}$ is not a classical trajectory as $K \neq \overline{p_3}/mc$. However, when $\mathrm{Im}\,z_3 \gg 1$, $K \simeq -\mathrm{Im}\,z_3 = \overline{p_3}/mc$ and $\overline{u^n(u^0)}$ coincide with the classical trajectory, the expression for the average value $\overline{u^3(u^0)}$ contains the integral which diverges and the trajectory for u^3 is absent.

6. Conclusion

In this paper we have found new exact solutions of the Klein–Gordon and Dirac equations in the electric field and the field of Redmond configuration (16), (22) and in the plane-wave field (32), (39), (45), (47). These solutions are coherent states (eigenfunctions corresponding to annihilation operators) either for some of the variables or for all variables. The main point in the construction of coherent states is the use of the 'null' plane. This allows us to find the unified method of constructing coherent states for relativistic particles. All the solutions form complete systems of functions.

If we consider the case when $A_n = 0$ in equations (32), (39), (45) and (47), we obtain the wavefunctions of coherent states of free particles.

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